FOURIER SERIES

A series of the form

$$\frac{1}{2}a_0 + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x) + \cdots + (a_n\cos nx + b_n\sin nx)$$

OR

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \quad \text{——— (1)}$$

is known to be a "Fourier series".

& a_0, a_n, b_n ($n = 1, 2, 3, \ldots$) which are constants independent of 'x' are known as "Fouries coefficients"

\# We have to expressed '$f(x)$' as a Fourier series. For this we have to find out Fourier Coefficients (a_0, a_n, b_n)

\# $f(x)$ will be called a periodic function 'α' if $f(x) = f(x+\alpha)$

$f(x)$ is said to be periodic function if $f(x+\alpha) = f(x)$ 'α' being the least +ve constant $\forall x$. & α is period

\# $\cos nx$ and $\sin nx$ are both periodic functions with period 2π. Therefore $f(x)$ is also periodic with period 2π.

In other words $f(x)$ is defined in the interval $[-\pi, \pi]$ / $[0, 2\pi]$

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$f(x) = f(x \pm 2n\pi)$ for $n \in I$

$\sin(x+2\pi) = \sin x$, $\operatorname{cosec}(x+2\pi) = \operatorname{cosec} x$

$\cos(x+2\pi) = \cos x$, $\sec(x+2\pi) = \sec x$

When $x \to x+2\pi$ The value of $\sin x$, $\cos x$, $\operatorname{cosec} x$, $\sec x$ remains unchanged.

ie 2π being least +ve constant

$||^{ly}$. $\tan(x+\pi) = \tan x$, $\cot(x+\pi) = \cot x$.

$x \longrightarrow x+\pi$, π being least +ve constant.

the value of $\tan x$, $\cot x$ remain unchanged.

$\therefore$ $\tan x$, $\cot x$ are periodic with period 'π'

e.g :- $\sin 3x = \sin(3x+2\pi) = \sin 3\left(x+\frac{2\pi}{3}\right)$

ie $x \to x+\frac{2\pi}{3}$, $\frac{2\pi}{3}$ being the least +ve constant, the value of $\sin 3x$ is unchanged

$||^{ly}$ $\cos 3x = \cos(3x+2\pi) = \cos 3\left(x+\frac{2\pi}{3}\right)$.

Generally $\sin nx$, $\cos nx$. are periodic functions with period '$\frac{2\pi}{n}$'

$\sin nx = \sin(nx+2\pi) = \sin n\left(x+\frac{2\pi}{n}\right)$

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$\sin\frac{\pi x}{l}$, $\cos\frac{\pi x}{l}$ are periodic functions with period ③

$$= \frac{2\pi}{\text{Coeff. of } 'x'} = \frac{2\pi}{\frac{\pi}{l}} = 2l$$

Some basic results on Integration

If m & n are integers, then

1. $$\int_{\alpha}^{\alpha+2\pi} \sin nx\,dx = \left[-\frac{\cos nx}{n}\right]_{\alpha}^{\alpha+2\pi} = -\frac{1}{n}\left[\cos nx\right]_{\alpha}^{\alpha+2\pi}$$

$$= -\frac{1}{n}\left[\cos n(\alpha+2\pi) - \cos n\alpha\right]$$

$$= -\frac{1}{n}\left[\cos n\alpha - \cos n\alpha\right]$$

$\boxed{\because \cos nx = \cos(nx+2\pi)}$

$$= 0$$

$$\therefore \int_{\alpha}^{\alpha+2\pi} \sin nx\,dx = 0 \;;\; n \neq 0$$

$$|||^{ly} \int_{\alpha}^{\alpha+2\pi} \cos nx\,dx = 0 \;;\; n \neq 0$$

2. $$\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx\,dx = 0 \quad \text{if } m \neq n$$

3. $$\int_{\alpha}^{\alpha+2\pi} \sin nx \cos mx\,dx = 0 \quad \text{if } m \neq n$$

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4. $\int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx \, dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} \sin 2nx \, dx$

$= -\frac{1}{4n}\left[\cos 2nx\right]_{\alpha}^{\alpha+2\pi} = -\frac{1}{4n}[0] = 0 \quad ; n \neq 0$

5. $\int_{\alpha}^{\alpha+2\pi} \sin^2 nx \, dx = \int_{\alpha}^{\alpha+2\pi} \left(\frac{1-\cos 2nx}{2}\right) dx$

$= \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_{\alpha}^{\alpha+2\pi} = \frac{1}{2}\left[\cancel{\alpha}+2\pi - \cancel{\alpha}\right]$

$\boxed{\because \left[\sin 2nx\right]_{\alpha}^{\alpha+2\pi} = 0}$

$= \frac{\cancel{2}\pi}{\cancel{2}} = \pi$

$\therefore \int_{\alpha}^{\alpha+2\pi} \sin^2 nx \, dx = \pi \quad ; n \neq 0$

6. $\int_{\alpha}^{\alpha+2\pi} \cos^2 nx \, dx = \int_{\alpha}^{\alpha+2\pi} \left(\frac{1+\cos 2nx}{2}\right) dx$

$= \frac{1}{2}\left[x + \frac{\sin 2nx}{2n}\right]_{\alpha}^{\alpha+2\pi} = \frac{1}{2}\left[\cancel{\alpha}+2\pi - \cancel{\alpha} + 0\right] = \pi$

$\therefore \int_{\alpha}^{\alpha+2\pi} \cos^2 nx \, dx = \pi \quad ; n \neq 0$

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7. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

8. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

9. $\int_a^b f(x)\,dx = \int_{a+2\pi}^{b+2\pi} f(x)\,dx$. ; f is periodic function of period 2π

10. $\int_{-\pi}^{\pi} f(x)\,dx = \int_{\alpha}^{\alpha+2\pi} f(x)\,dx = \int_0^{2\pi} f(x)\,dx$

11. $\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} f(c+x)\,dx$. ; a, b, c are any numbers.

12. when $f(x)$ is an odd functions

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$

As $\cos nx$ is an even function

$\therefore$ $f(x)\cos nx$ will be an odd function

$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$

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